Chapter - 8

Introduction to Trigonometry

Trigonometric Ratios

Introduction

Trigonometry is the study of relationships between sides and angles of a triangle. The word trigonometry is derived from the Greek words 'tri' (meaning three), 'gon' meaning sides and 'metron' (meaning measure). Application of Trigonometry

The fundamental of trigonometry is used to design bridges and build structures.

Trigonometry is used to measure the height of a building or a mountain.

Trigonometry is used by surveyors to measure distances and angles between points on land.
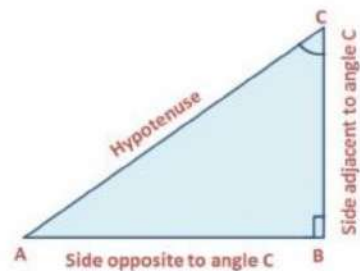
Trigonometric Ratios

In a right triangle, one of the angles is 90° and the other two angles are less than 90°. The names of each side of a right triangle are,

Hypotenuse	Base	Perpendicular
• The longest side of a right triangle is hypotenuse. It is the side opposite to the right angle of the triangle.	• The side of the right triangle on which it stands is known as the base of the triangle.	• The side which is perpendicular to the base is the perpendicular.

∠A	Reference Angle
BC	Perpendicular, the Side opposite to ∠A
AC	Hypotenuse, opposite side to the right angle
AB	Adjacent Side to ∠A or Base

| ∠C	Reference Angle
AB	Perpendicular, the Side opposite to ∠C
AC	Hypotenuse, opposite side to the right angle
BC	Adjacent Side to ∠C or Base

Trigonometric Ratios of an acute angle in a right triangle is the relationship between the angle and the length of its sides.

The trigonometric ratios of the ∠ A in right triangle ABC are defined as follows:

$$\text{sine or sin of } \angle A = \frac{\text{Side opposite to } \angle A}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine or cos of } \angle A = \frac{\text{Side adjacent to } \angle A}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent or tan of } \angle A = \frac{\text{Side opposite to } \angle A}{\text{Side adjacent to } \angle A} = \frac{BC}{AB}$$
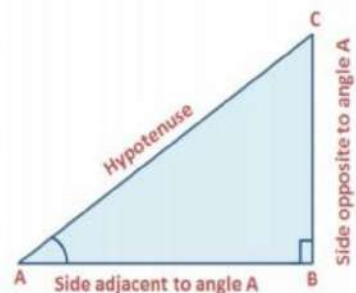
The ratios defined above are abbreviated as sin A, cos A, tan A

$$\text{cosecant or cosec of } \angle A = \frac{1}{\text{sine of } \angle A} = \frac{\text{hypotenuse}}{\text{Side opposite to } \angle A} = \frac{AC}{BC}$$

$$\text{secant or sec of } \angle A = \frac{1}{\text{cosine of } \angle A} = \frac{\text{hypotenuse}}{\text{Side adjacent to } \angle A} = \frac{AC}{AB}$$

$$\text{cotangent or cot of } \angle A = \frac{1}{\text{tangent of } \angle A} = \frac{\text{Side adjacent to } \angle A}{\text{Side opposite to } \angle A} = \frac{AB}{BC}$$

The ratios cosec A, sec A, and cot A are respectively the reciprocals of the ratios sin A, cos A and tan A.

$$\tan A = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin A}{\cos A} \text{ and } \cot A = \frac{\cos A}{\sin A}$$

The symbol sin A is used as an abbreviation for the 'sine of the angle A' and it is not the product of 'sin' and A.
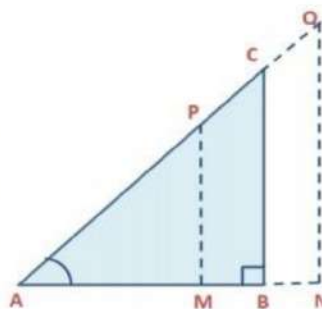
> The values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.

Let ABC be a right triangle. Let P be a point on hypotenuse AC and Q be a point on AC extended.

Draw PM perpendicular to AB and QN perpendicular to AB extended.

In Δ PAM and Δ CAB

∠PAM = ∠CAB	Common Angle
∠AMP = ∠ABC	90⁰
ΔPAM ~ΔCAB	By AA Similarity Criterion

By the property of similar triangles, corresponding sides of the triangle are proportional.

$$\therefore \frac{AM}{AB} = \frac{AP}{AC} = \frac{MP}{BC} \rightarrow \text{Eq 1}$$

From Eq 1 we

$$\frac{AP}{AC} = \frac{MP}{BC} \Rightarrow \frac{MP}{AP} = \frac{BC}{AC} = \sin A$$

Similarly, $\frac{AM}{AP} = \frac{AB}{AC} = \cos A, \frac{MP}{AM} = \frac{BC}{AB} = \tan A$

We see that the trigonometric ratios of ∠A in ΔPAM are same as those of ∠A in ΔCAB.
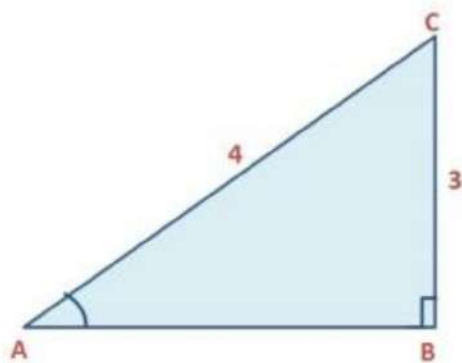
Example: If $\sin A = \dfrac{3}{4}$, calculate cos A and tan A.

Now, $\sin A = \dfrac{3}{4} \Rightarrow \dfrac{BC}{AC} = \dfrac{3}{4}$

So, we draw a ΔABC such that, BC = 3 units and AC = 4 cm. Using Pythagoras theorem, we get

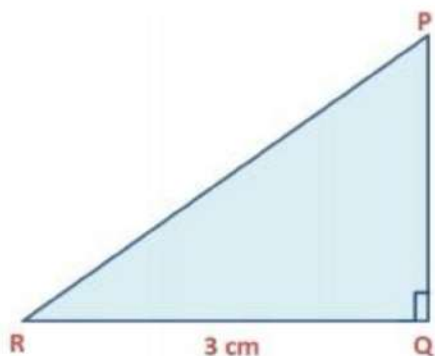$AC^2 = AB^2 + BC^2 \Rightarrow 4^2 = AB^2 + 3^2 \Rightarrow AB^2 = 16 - 9 = 7$

$AB = \sqrt{7}$

Here, AB = $\sqrt{7}$ units, BC = 3 units and AC = 4 cm.

$\cos A = \dfrac{AB}{AC} = \dfrac{\sqrt{7}}{4}$ and $\tan A = \dfrac{BC}{AB} = \dfrac{3}{\sqrt{7}}$

Example: In ΔPQR right-angled at Q, QR = 3 cm and PR – PQ = 1 cm. Determine the values of sin R, cos R, and tan R.

In ΔPQR, ∠Q = 90° and QR = 3 cm

Also, PR – PQ = 1 cm → Eq 1

Using Pythagoras theorem, we get

$PR^2 = PQ^2 + QR^2 \Rightarrow QR^2 = PR^2 - PQ^2$

$\Rightarrow PR^2 - PQ^2 = 9$ (∵ QR = 3 cm)

$9 = (PR + PQ)(PR - PQ)$ [∵ $PR^2 - PQ^2 = (PR + PQ)(PR - PQ)$]

Now, PR – PQ = 1 cm

$9 = (PR + PQ) \times 1 \Rightarrow PR + PQ = 9$ →Eq 2

Adding Eq 1 and Eq 2 we get,

$$PR - PQ + PR + PQ = 9 + 1$$
$$2\,PR = 10 \Rightarrow PR = 5 \text{ cm}$$
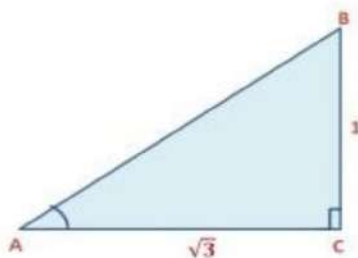
Substituting PR = 5 cm in Eq 1 we get,

$$5 - PQ = 1 \Rightarrow PQ = 4 \text{ cm}$$

Now, $\sin R = \dfrac{PQ}{PR} = \dfrac{4}{5}$, $\cos R = \dfrac{QR}{PR} = \dfrac{3}{5}$ and $\tan R = \dfrac{PQ}{QR} = \dfrac{4}{3}$

Example: In ΔABC right-angled at C, if $\tan A = \dfrac{1}{\sqrt{3}}$, find the value of sin A cos B + cos A sin B.

Now, $\tan A = \dfrac{1}{\sqrt{3}} \Rightarrow \dfrac{BC}{AC} = \dfrac{1}{\sqrt{3}}$

First, we draw ΔABC right angled at C such that, BC = 1 unit and AC = √3 unit. Using Pythagoras theorem, we get

$AB^2 = BC^2 + AC^2 \Rightarrow AB^2 = 1^2 + (\sqrt{3})^2$

$\Rightarrow AB^2 = 3 + 1 = 4$

AB = 2 units

With reference to ∠A, we have, Base = AC = √3, Hypotenuse AB = 2 units, Perpendicular BC = 1 unit

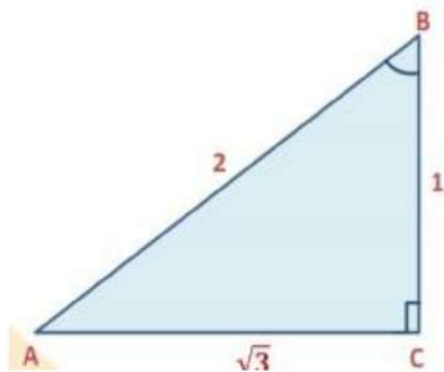$\sin A = \dfrac{BC}{AB} = \dfrac{1}{2}$, $\cos A = \dfrac{AC}{AB} = \dfrac{\sqrt{3}}{2}$

With reference to ∠B we have,

Base BC = 1, Hypotenuse AB = 2 units,

Perpendicular AC = √3 unit

$\sin B = \dfrac{AC}{AB} = \dfrac{\sqrt{3}}{2}$, $\cos B = \dfrac{BC}{AB} = \dfrac{1}{2}$

$\therefore \sin A \cos B + \cos A \sin B = \dfrac{1}{2} \times \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{1}{4} + \dfrac{3}{4} = \dfrac{4}{4} = 1$
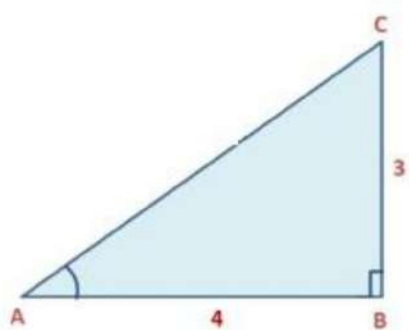

Example: If 3 cot A = 4, check whether $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A$ or not.

Now, 3 cot A = 4 $\Rightarrow \cot A = \dfrac{4}{3} \Rightarrow \dfrac{B}{P} = \dfrac{4}{3}$

First, we draw ΔABC right angled at B such that, BC = 3 units and AB = 4 units.

Using Pythagoras theorem, we get

AC² = AB² + BC² ⇒ AC² = 4² + 3² ⇒ AC² = 16 + 9 = 25

AC = $\sqrt{25}$ ⇒ AC = 5units

tan A = $\dfrac{1}{cotA} = \dfrac{3}{4}$, cos A = $\dfrac{AB}{AC} = \dfrac{4}{5}$ $and\ sinA = \dfrac{BA}{AC} = \dfrac{3}{5}$

$Now. LHS = \dfrac{1 - tan^2 A}{1 + tan^2 A} = \dfrac{1 - (\frac{3}{4})^2}{1 + (\frac{3}{4})^2} = \dfrac{1 - \frac{9}{16}}{1 + \frac{9}{16}} = \dfrac{\frac{16-9}{16}}{\frac{16+9}{16}} = \dfrac{7/16}{25/16} = \dfrac{7}{5}$

$RHS = cos^2 A - sin^2 A = (\dfrac{4}{5})^2 - (\dfrac{3}{5})^2 = \dfrac{16}{25} - \dfrac{9}{25} = \dfrac{7}{25}$

LHS = RHS

Example: If m cot A = n, find the value of $\dfrac{m sinA - n cosA}{n ocsA + m sinA}$

Now, m cot A = n ⇒ m. $\dfrac{1}{tanA}$ = n ⇒ tan A = $\dfrac{m}{n}$

$\dfrac{m sinA - n cosA}{n cosA + m sinA}$, Dividing both numerator and denominator by cos A

$= \dfrac{m\frac{sinA}{CosA} - n\frac{cosA}{cosA}}{n\frac{cosA}{cosA} + m\frac{sinA}{cosA}} = \dfrac{m\frac{m}{n} - n}{n + m.\frac{m}{n}} (\because tanA = \dfrac{m}{n})$

$$\frac{\frac{m^2-n^2}{n}}{\frac{n^2+m^2}{n}} = \frac{m^2-n^2}{m^2+n^2}$$

## Trigonometric Ratios of Some Specific Angles

Trigonometric Ratios of Some Specific Angles

We will find the trigonometric ratios of some specific angles by a geometrical method. It is important to understand the trigonometric ratios of these angles as they help us to solve trigonometric problems easily.

Trigonometric Ratios of 45°

Let ΔABC be a right-angled triangle, in which ∠B = 90° and ∠A = 45°, then the third angle will also be equal to 45° , i.e. ∠C = 45°

So, BC = AB = a (∵ sides opposite to equal angles of a triangle are also equal)
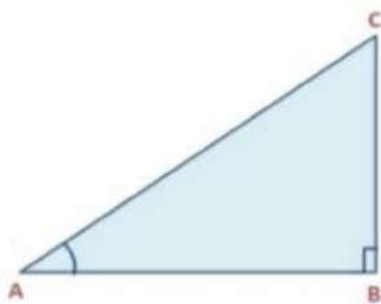
Then by Pythagoras theorem, $AC^2 = AB^2 + BC^2 = a^2 + a^2$

$AC = \sqrt{2}a$
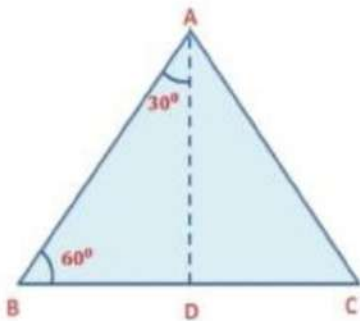
Using the definition of trigonometric ratios we have,

$\sin 45° = \dfrac{\textit{side opposite to angle 45}}{\textit{hypotenuse}} = \dfrac{BC}{AC} = \dfrac{a}{\sqrt{2}a} = \dfrac{1}{\sqrt{2}}$

$\cos 45° = \dfrac{\textit{side adjacent to angle 45}}{\textit{hypotenuse}} = \dfrac{AB}{AC} = \dfrac{a}{\sqrt{2}a} = \dfrac{1}{\sqrt{2}}$

$\tan 45° = \dfrac{\textit{side opposite to angle 45}}{\textit{side adjacent to angle 45}} = \dfrac{BC}{AB} = \dfrac{a}{a} = 1$

$\csc 45° = \dfrac{1}{\sin 45°} = \sqrt{2}$	
$\sec 45° = \dfrac{1}{\cos 45°} = \sqrt{2}$	
$\cot 45° = \dfrac{1}{\tan 45°} = 1$	

Trigonometric Ratios of 30° and 60°

Let ABC be an equilateral triangle. Then $\angle A = \angle B = \angle C = 60°$. Draw the perpendicular AD from A to side BC.

Now, $\Delta ABD \cong \Delta ACD$ (By RHS Congruence Rule)

Therefore, BD = DC and $\angle BAD = \angle CAD$ (CPCT)

So, ΔABD is a right triangle, right-angled at D with

$\angle BAD = 30°$ and $\angle ABD = 60°$

Let AB = 2a then, $BD = \dfrac{1}{2} BC = a$

$AD^2 = AB^2 - BD^2 \Rightarrow AD^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2$

$AD = \sqrt{3}a$
Here, AB = 2a, BD = a and AD = $\sqrt{3}a$

For angle 30°

|	
1) $\sin 30° = \dfrac{BD}{AB} = \dfrac{a}{2a} = \dfrac{1}{2}$	2) $\cos 30° = \dfrac{AD}{AB} = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$
3) $\tan 30° = \dfrac{BD}{AD} = \dfrac{a}{\sqrt{3}a} = \dfrac{1}{\sqrt{3}}$	4) $\operatorname{cosec} 30° = \dfrac{1}{\sin 30°} = 2$
5) $\sec 30° = \dfrac{1}{\cos 30°} = \dfrac{2}{\sqrt{3}}$	6) $\cot 30° = \dfrac{1}{\tan 30°} = \sqrt{3}$

For angle 60°

|	
1) $\sin 60° = \dfrac{AD}{AB} = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$	2) $\cos 60° = \dfrac{BD}{AB} = \dfrac{a}{2a} = \dfrac{1}{2}$
3) $\tan 60° = \dfrac{AD}{BD} = \dfrac{\sqrt{3}a}{a} = \sqrt{3}$	4) $\operatorname{cosec} 60° = \dfrac{1}{\sin 60°} = \dfrac{2}{\sqrt{3}}$
5) $\sec 60° = \dfrac{1}{\cos 60°} = 2$	6) $\cot 60° = \dfrac{1}{\tan 60°} = \dfrac{1}{\sqrt{3}}$

Trigonometric Ratios of 0º and 90º

In ΔABC,

• As ∠A gets smaller and smaller, the length of the side BC decreases.

• Point C comes closer to point B and finally when ∠A becomes very close to 0º, AC becomes almost the same as AB.

Let's learn this with the help of a diagram.
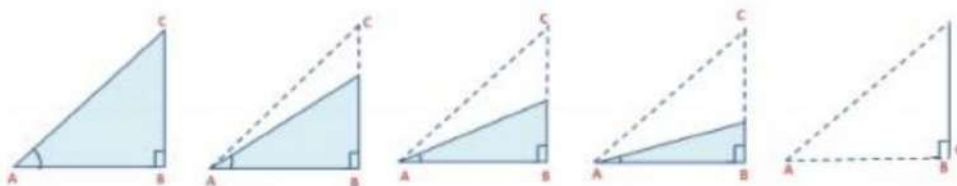
When ∠A is very close to $0°$, Side BC is very close to 0
$\therefore \sin A = \dfrac{BC}{AC}$ is very close to 0
Side AC is nearly as same as AB
$\therefore \cos A = \dfrac{AB}{AC}$ is very close to 1
$\sin 0° = 0$ and $\cos 0° = 1$

Now we will what happens when ∠ A becomes larger and larger till it becomes 90º.
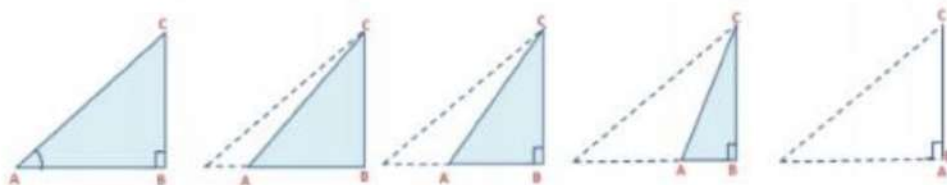
• As ∠A increases, ∠C decreases.

• The length of the side AB decreases.

• Point A comes closer to point B.

When ∠A is very close to 90°, ∠C is very close to 0°
Side AC is nearly as same as BC
∴ $\sin A = \dfrac{BC}{AC}$ is very close to 1
Side AB is very close to 0
∴ $\cos A = \dfrac{AB}{AC}$ is very close to 0
$\sin 90° = 1$ and $\cos 90° = 0$

∠A	0°	30°	45°	60°	90°
sin A	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1
cos A	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0
tan A	0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\dfrac{2}{\sqrt{3}}$	1
sec A	1	$\dfrac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\dfrac{2}{\sqrt{3}}$	0

The value of sin A increases from 0 to 1 and cos A decreases from 1 to 0, where $0° \leq A \leq 90°$

Example: If $2\cos 2\theta = \sqrt{3}$, find the value of θ.

Here, $2\cos 2\theta = \sqrt{3}$

$\cos 2\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \cos 2\theta = \cos 30^0$

$\Rightarrow 2\theta = 30^0 \Rightarrow \theta = 15^0$

Example: Evaluate

$8\sqrt{3}\csc^2 30°\sin 60°\cos 60°\cos^2 45°\sin 45°\tan 30°\csc^3 45°$

$= 8\sqrt{3}(\csc 30^0)^2 \sin 60^0 \cos 60^0 (\cos 45^0)^2 \sin 45^0 \tan 30^0 (\csc 45^0)^3$

$= 8\sqrt{3} \times (2)^2 \times \dfrac{\sqrt{3}}{2} \times \dfrac{1}{2} \times (\sqrt{2})^2 \times \dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{3}} \times (\sqrt{2})^3 = 8\sqrt{3} \times 4 \times \dfrac{\sqrt{3}}{4} \times \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{3}} \times 2\sqrt{2}$

$= \dfrac{8\sqrt{3} \times 4}{4} = 8\sqrt{3}$

Example: If $\sin (A - B) = \dfrac{1}{2}$, $\cos (A + B) = \dfrac{1}{2}$, $0^0 \le A + B \le 90^0$, $A > B$, find the values of A and B.

Here, $\sin (A - B) = \dfrac{1}{2} \Rightarrow \sin (A - B) = \sin 30^0 \; (\because \sin 30^0 = \dfrac{1}{2})$

$\Rightarrow A - B = 30^0 \rightarrow$ Eq 1

$\cos (A + B) = \dfrac{1}{2} \Rightarrow \cos(A + B) = \cos 60^0 \; (\because \cos 60^0 = \dfrac{1}{2})$

$\Rightarrow A + B = 60^0 \rightarrow$ Eq 2

On adding Eq 1 and Eq 2 we get,

$A - B + A + B = 30^0 + 60^0 \Rightarrow 2A = 90^0 \Rightarrow A = 45^0$

Putting the value of A in Eq 1 we get,

$45^0 - B = 30^0 \Rightarrow B = 45^0 - 30^0 \Rightarrow B = 15^0$

Therefore, $A = 45^0$ and $B = 15^0$

Example: If $\tan \theta = 1$ and $\sin \varphi = \dfrac{1}{\sqrt{2}}$, find the value of $\cos(\theta + \varphi)$ and $\sin(\theta + \varphi)$, where θ and φ are both acute angles.

Here, $\tan \theta = 1 \Rightarrow \tan \theta = \tan 45^0 \Rightarrow \theta = 45^0$ ($\because \tan 45^0 = 1$) and $\sin \varphi = \dfrac{1}{\sqrt{2}} \Rightarrow$

$\sin \varphi = \sin 45^0 \Rightarrow \varphi = 45^0$ ($\because \sin 45^0 = \dfrac{1}{\sqrt{2}}$)

Now, $\theta = 45^0$ and $\varphi = 45^0$

$\theta + \varphi = 45^0 + 45^0 = 90^0$

$\cos(\theta + \varphi) = \cos 90^0 = 0$

$\sin(\theta + \varphi) = \sin 90^0 = 1$

Example: State whether the following statements are true or false.

Justify your answer.

i) $\sin(A + B) = \sin A + \sin B$

Let $A = 45^0$ and $B = 45^0$

$\sin(A + B) = \sin(45^0 + 45^0) = \sin 90^0 = 1$

$\sin A + \sin B = \sin 45° + \sin 45°\; \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \dfrac{1+1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

$\sin(A + B) \neq \sin A + \sin B$

Hence, the given statement is false.

ii) The value of $\cos A$ increases as A increases.

Consider the table given below

∠A	0°	30°	45°	60°	90°
cos A	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0

So, we see that the value of cos A decreases as A increases. Hence the statement is false.

Trigonometric Ratios of Complementary Angles

Trigonometric Ratios of Complementary Angles

Two angles are said to be complementary if their sum is equal to 90°.

Consider the right-angled Δ ABC, right-angled at B.

As, ∠ B = 90°, then ∠ A + ∠ C = 90°

(∵ In a triangle ∠ A + ∠ B + ∠ C = 180°)

Therefore, ∠ A and ∠ C form a pair of complementary angles.
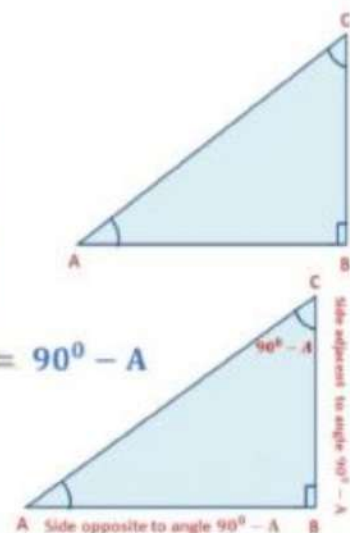
We have,

$\sin A = \dfrac{BC}{AC}$	$\cos A = \dfrac{AB}{AC}$
$\tan A = \dfrac{BC}{AB}$	$\operatorname{cosec} A = \dfrac{AC}{BC}$
$\sec A = \dfrac{AC}{AB}$	$\cot A = \dfrac{AB}{BC}$

Now, we will write the trigonometric ratios for ∠C = 90° − A

(For convenience ∠ A is written as A)

Now,

$\sin C = \sin(90^0 - A) = \dfrac{AB}{AC}$	$\cos C = \cos(90^0 - A) = \dfrac{BC}{AC}$
$\tan C = \tan(90^0 - A) = \dfrac{AB}{BC}$	$\operatorname{cosec} C = \operatorname{cosec}(90^0 - A) = \dfrac{AC}{AB}$
$\sec C = \sec(90^0 - A) = \dfrac{AC}{BC}$	$\cot C = \cot(90^0 - A) = \dfrac{BC}{AB}$

If we compare the two sets of ratios we get,

$\sin(90^0 - A) = \cos A$	$\cos(90^0 - A) = \sin A$
$\tan(90^0 - A) = \cot A$	$\operatorname{cosec}(90^0 - A) = \sec A$
$\sec(90^0 - A) = \operatorname{cosec} A$	$\cot(90^0 - A) = \tan A$

Example: Express cosec 58^0 + tan 88^0 in terms of T- ratios of angles 0^0 to 45^0.

Here, cosec 58^0 + tan 88^0 = cosec $(90^0 - 32^0)$ + tan $(90^0 - 2^0)$

= sec 32^0 + cot 2^0

$\{\because$ cosec $(90^0 - A) = $ sec A and tan $(90^0 - A) = $ cot A$\}$

Example: Evaluate $\dfrac{\tan 15}{\cot 75} + \dfrac{\sin 25}{\cos 65}$

$$\dfrac{\tan 15}{\cot 75} + \dfrac{\sin 25}{\cos 65} = \dfrac{\tan(90 - 75)}{\cot 75} + \dfrac{\sin(90 - 65)}{\cos 65}$$

$$= \dfrac{\cos 75}{\cot 75} + \dfrac{\cos 65}{\cos 65} (\because \tan(90 - A) = \cot A \, and \sin(90 - A) = cas A)$$

$$= 1 + 1 = 2$$

Example: Find the value of (tan 1^otan 2^otan 3° … … .tan 89°)

(REFERENCE: NCERT)

Here, tan 1^otan 2^otan 3° … … .tan 89°

=tan $(90^o - 89^o)$ tan $(90^o - 88^o)$ tan $(90^o - 87^o)$ … tan 45° … .

tan 87^otan 88^otan 89°

=cot 89°cot 88°cot 87° ... tan 45° ... .tan 87°tan 88°tan 89°

$$\{\because \tan (90° - \theta) = \cot \theta\}$$

= (cot 89°tan 89°) (cot 88°tan 88°) (cot 87°tan 87°) ... ...

(cot 46°tan46°) tan 45°

(∵ tan θ. cot θ = 1 and tan 45° = 1)
= 1 × 1 × 1 ... ... × 1 = 1

Example: Evaluate

$$\frac{tan50 + swec50}{cot40 + cosec40} + \cos 40° \ cosec \ 50°$$

Now, $\frac{tan50 + sec50}{cot40 + cosec40} + cos40 cosec50$

$$= \frac{tan(90 - 40) + sec(90 - 40)}{cot40 + cose40 + cos40cosec(90 - 40)}$$

$$\{\because \tan (90° - \theta) = \cot\theta\}$$

$$= \frac{cot40 + cosec40}{cot40 + cosec40} + \cos 40° \ \sec 40° \ \{\because cosec \ (90° - \theta) = \sec\theta\}$$

$$= 1 + \cos 40° \times \frac{1}{cos40} = 1 + 1 = 2$$

Example: If cos (A – B) = cos A cos B + sin A sin B, find the value of

i) cos15°

We know, cos (A – B) = cos A cos B + sin A sin B → Eq 1

Let A = 45° and B = 30°

cos 15° = cos (45° − 30°)

cos (45° − 30°) = cos 45° cos 30° + sin 45° sin 30° (Using Eq 1)

$$\frac{1}{\sqrt{2}} X \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} X \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$(\because \cos 45° = \frac{1}{\sqrt{2}}, \cos 30° = \frac{\sqrt{3}}{2}, \sin 45° = \frac{1}{\sqrt{2}}, \sin 30° = \frac{1}{2})$

$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

ii) sin 75°

$\sin 75° = \sin(90° − 15°) = \cos 15° \{\because \sin(90° − \theta) = \cos \theta\}$

We know, $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$
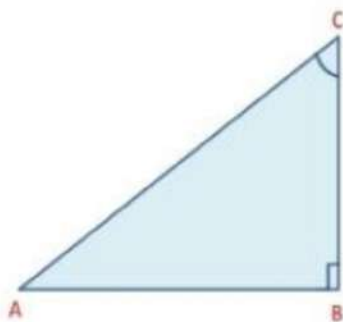
Trigonometric Identities

Trigonometric Identities

An equation is called an identity when it is true for all values of the variables involved.

Similarly, an equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle (s) involved

Consider ΔABC, right-angled at B.
$AB^2 + BC^2 = AC^2 \rightarrow$ Eq 1 (By Pythagoras Theorem)

Dividing each term of Eq 1 by AC^2 we get,

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = 1$$

$(\cos A)^2 + (\sin A)^2 = 1$ $(\because \cos A = \frac{AB}{AC}$ and $\sin A = \frac{BC}{AC})$

$$\cos^2 A + \sin^2 A = 1$$

This is true for all A such that $0° \leq A \leq 90°$. So this is a trigonometric identity.

Now, we divide Eq 1 by AB^2 to get,

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$1 + \tan^2 A = \sec^2 A$ $(\because \tan A = \frac{BA}{AB}$ and $\sec A = \frac{AC}{AB})$

This equation is true for A=0°, but tan A and sec A are not defined for A = 90°. So, this is true for all A such that $0° \leq A < 90°$

Dividing Eq 1 by BC^2 we get,

$$\frac{AB^2}{BA^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

$\cot^2 A + 1 = \text{cosec}^2 A$ $(\because \cot A = \frac{AB}{BC}$ and $\text{cosec } A = \frac{AC}{BC})$

cosec A and cot A are not defined for A = 0°. So, this is true for all A such that 0°

Example: Prove that $\dfrac{sinA - cosA + 1}{sinA + cosA - 1} = \dfrac{1}{secA - tanA}$, using identity sec²A = 1 + tan²A

(REFERENCE: NCERT)

LHS = $\dfrac{sinA - cosA + 1}{sinA + cosA - 1} = \dfrac{\frac{sinA}{CosA} - \frac{cosA}{cosA} + \frac{1}{coaA}}{\frac{sinA}{cosA} + \frac{cosA}{cosA} - \frac{1}{cosA}}$

Dividing both numerator and denominator by cos A we get,

$= \dfrac{tanA - 1 + secA}{tanA + 1 - secA} \Rightarrow \dfrac{(tanA + secA) - 1}{tanA - secA + 1}$

Multiplying and Dividing by (tan A – sec A)

$\dfrac{[(tanA + secA) - 1][tanA - secA]}{[(tanA - SecA) + 1][tanA - secA]}$

{ ∵ (a − b)(a + b) = a² − b²}

$= \dfrac{-1 - tanA + secA}{[(tanA - secA) + 1][tanA - secA]} (\because tan^2 A - sec^2 A = -1)$

$= \dfrac{-(\%u2212(1 + tanA\%u2212secA))}{[(tanA - secA) + 1][tanA - secA]}$

$\dfrac{-1}{tanA - secA} = \dfrac{1}{secA - tanA} = RHS$

Example: If a cos A − b sin A = x and a sin A + b cos A = y, prove that a² + b² = x² + y².

Here, a cos A − b sin A = x → Eq 1
a sin A + b cos A = y → Eq 2

Squaring and then adding Eq 1 and Eq 2 we get,
x² + y² = (a cos A − b sin A)² + (a sin A + b cos A)²

x² + y² = a²cos²A + b²sin²A − 2ab sin Acos A + a²sin²A + b²cos²A + 2ab sin Acos A {∵ (a ± b)² = a² + b² ± 2ab}

$x^2 + y^2 = a^2(\cos^2A + \sin^2A) + b^2(\sin^2A + \cos^2A)$

$x^2 + y^2 = a^2 + b^2 \ (\because \sin^2A + \cos^2A = 1)$

Example: If tan A + sin A = m and A − sin A = n, then show that $(m^2 - n^2)^2 = 16\,mn$ or $(m^2 - n^2) = 4\sqrt{mn}$

Here, tan A + sin A = m → Eq 1

tan A − sin A = n → Eq 2

Adding Eq 1 and Eq 2 we get,

$2 \tan A = m + n \Rightarrow \tan A = \dfrac{m+n}{2}$

$\therefore \cot A = \dfrac{1}{\tan A} = \dfrac{2}{m+n} \to$ Eq 3

Subtracting Eq 2 from Eq 1 we get,

$2 \sin A = m - n \Rightarrow \sin A = \dfrac{m-n}{2}$

$\therefore \operatorname{cosec} A = \dfrac{1}{san A} = \dfrac{2}{m-n} \to$ Eq 4

We know that, $\operatorname{cosec}^2A - \cot^2A = 1$

Using Eq 3 and Eq 4 we get,

$$\left(\dfrac{2}{m-n}\right)^2 - \left(\dfrac{2}{m+n}\right)^2 = 1 \Rightarrow \dfrac{4}{(m-n)^2} - \dfrac{4}{(m+n)^2} = 1$$

$$4\left[\dfrac{1}{(m-n)^2} - \dfrac{1}{(m+n)^2}\right] = 1 \Rightarrow 4\left[\dfrac{(m+n)^2 - (m-n)^2}{(m-n)^2(m+n)^2}\right] = 1$$

$$\Rightarrow 4\left[\dfrac{(m^2+n^2+2mn) - (m^2+n^2+2mn)}{(m-n)^2(m+n)^2}\right] = 1 \because (a \pm b)^2 = a^2 + b^2 \pm 2ab$$

$$\Rightarrow 4\left[\dfrac{4mn}{(m-n)^2(m+n)^2}\right] = 1 \Rightarrow \dfrac{16mn}{[(m-n)(m+n)]^2} = 1$$

$\Rightarrow 16mn = (m^2 - n^2)^2 \{\because (a - b)(a + b) = a^2 - b^2\}$

$\Rightarrow (m^2 - n^2)^2 = 16mn$ or $(m^2 - n^2) = 4\sqrt{mn}$

Example: Prove the identity

$$(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$$

(REFERENCE: NCERT)

$\text{LHS} = (\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2$

$= \sin^2 A + \text{cosec}^2 A + 2\sin A \text{ cosec } A + \cos^2 A + \sec^2 A + 2\cos A \sec A$

$\{\because (a + b)^2 = a^2 + b^2 + 2ab\}$

$= \sin^2 A + \cos^2 A + \text{cosec}^2 A + \sec^2 A + 2\sin A \text{ cosec } A + 2\cos A \sec A$

$= 1 + (1 + \cot^2 A) + (1 + \tan^2 A) + 2\sin A \times \dfrac{1}{sinA} + 2cosAX\dfrac{1}{cosA}$

$[\because \sin^2 A + \cos^2 A = 1, \text{cosec}^2 A = 1 + \cot 2A, \text{cosec A} \quad = \dfrac{1}{sonA} and secA \dfrac{1}{cosA}]$

$= 1 + 1 + \cot^2 A + 1 + \tan^2 A + 2 + 2 \Rightarrow 7 + \tan^2 A + \cot^2 A$